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Attitude Regulation About a Fixed Rotation Axis

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The attitude control problem of rotating a rigid body from its current attitude to a desired attitude such that the instantaneous rotation axis is aligned with an externally defined axis of rotation is considered. The key idea is to factor the attitude quaternion into rotations parallel and perpendicular to the desired rotation axis. State feedback and output feedback control strategies are designed to minimize the perpendicular component. Simulation results are provided to illustrate the salient features of the approach.

Nomenclature

$\boldsymbol{e}$	=	unit quaternion representation of the attitude error, $\boldsymbol{q}^{d*}\boldsymbol{q}$
$\boldsymbol{e}_{\parallel}$	=	component of $\boldsymbol{e}$ that is parallel to $\boldsymbol{u}$
$\boldsymbol{e}_{\perp}$	=	component of $\boldsymbol{e}$ that is perpendicular to $\boldsymbol{u}$
J	=	inertia matrix of the spacecraft
p_0	=	scalar part of the unit quaternion $\boldsymbol{p}$
$\hat{\boldsymbol{p}}$	=	vector part of the unit quaternion $\boldsymbol{p}$
$\boldsymbol{q}$	=	unit quaternion representation of the attitude of the spacecraft
$\boldsymbol{q}^d$	=	unit quaternion representation of the desired attitude of the spacecraft
$\boldsymbol{q}^*$	=	conjugate of $\boldsymbol{q}$, $(-\hat{\boldsymbol{q}}^T, q_0)^T$
$\boldsymbol{u}$	=	desired axis of rotation
θ	=	$2\tan^{-1}(\boldsymbol{u}^T\hat{\boldsymbol{e}}/e_0)$
$\boldsymbol{\tau}$	=	torque applied to the spacecraft
$\boldsymbol{\omega}$	=	angular velocity of the spacecraft
$\bar{\boldsymbol{\omega}}$	=	$(\boldsymbol{\omega}^T, 0)^T$
$\boldsymbol{\omega}_{\parallel}$	=	angular velocity associated with $\boldsymbol{e}_{\parallel}$, $\dot{\theta}\boldsymbol{u}$
$\boldsymbol{\omega}_{\perp}$	=	angular velocity associated with $\boldsymbol{e}_{\perp}$
$\mathbf{1}$	=	unit quaternion identity, $(0, 0, 0, 1)^T$

I. Introduction

RIGID-BODY attitude regulation is a problem that has generated much interest. A great deal of work has been done in this area, including attitude tracking,[1] inverse optimal stabilization,[2] approximate solutions to the optimal attitude control problem,[3] and H_∞ suboptimal control.[4,5] These results assume that the spacecraft angular velocity is known. A good approximation of the spacecraft angular velocity is often not available. Passivity-based control has been derived to regulate the attitude of a spacecraft,[6,7] without velocity information.

Sometimes the path taken to the final orientation is as important as is the final orientation. For example, a problem that has attracted attention lately is the spacecraft formation control problem.[8] The attitude formation control problem described in Refs. 8 and 9 requires that the spacecraft constellation perform a maneuver such that the attitude of the individual spacecraft remain synchronized. To accomplish this, the spacecraft must rotate about axes of rotation that are parallel to each other.

This type of maneuver leads to an interesting spacecraft attitude control problem. If the spacecraft deviates from the desired axis of rotation, then the objective of the control law is to realign the axis of rotation, with the other spacecraft, as well as synchronize the angle of rotation along that direction. Attitude synchronization has been discussed in Ref. 8. The objective of this paper is to present a model-independent attitude control law that facilitates the types of formation maneuvers discussed earlier, that is, to rotate a rigid body, as closely as possible, about an externally assigned axis of rotation.

Our objective is in contrast to the well-known problem of eigenaxis rotation where the goal is to rotate through the shortest possible angle.[10,11] If the spacecraft if performing a rest-to-rest maneuver, and it deviates from the original eigenaxis, the control objective is still to maneuver the spacecraft to its final attitude through the shortest possible angle. Therefore, unlike the problem addressed in this paper, the eigenaxis may change in time, due to disturbances or model uncertainty.

Current strategies for eigenaxis rotations use a combination of feedback control and feedforward control terms. Wie et al.[12] used feedback on the quaternion error combined with feedforward of the gyroscopic term to implement eigenaxis rotations via quaternion regulation. These regulation results were extended to the tracking problem by Weiss.[13] The tracking results are also model dependent and require cancellation of the gyroscopic term. Some effort has been made to derive a model-independent approach to quaternion regulation via eigenaxis rotations. Cristi et al.[14] developed adaptive control strategies to approximate the control found in Ref. 12. Output adaptive control is used in Ref. 15 to approximate an inertia matrix.

Our approach it to consider a rest-to-rest maneuver and to factor the required rotation into a rotation about $\boldsymbol{u}$, where $\boldsymbol{u}$ is an arbitrary unit vector fixed in the inertial frame, followed by a rotation about an axis perpendicular to $\boldsymbol{u}$. The control law then damps out the rotation perpendicular to $\boldsymbol{u}$.

A paper that is related to our approach is Ref. 16, which addresses the problems of spin stabilization about an arbitrary axis defined in the body frame. However, the approach in Ref. 16 differs from our approach in several aspects. First, the rotation axis in Ref. 16 is fixed in the body frame, in contrast to our approach, where $\boldsymbol{u}$ is fixed in the inertial frame. Second, in Ref. 16, the rotation axis must be known a priori, because the control strategy depends explicitly on the rotation axis. On the other hand, in our paper, $\boldsymbol{u}$ is an input to the controller, which is designed independent of $\boldsymbol{u}$. Finally, the control strategy in Ref. 16 is model dependent in that it requires knowledge of J, where our result is independent of J.

The paper is organized as follows. In Sec. II, the mathematical notation and models used in the paper will be introduced. In addition, a precise statement of the rotation axis attitude control problem will be given. In Sec. III, we present two control strategies for rotation axis attitude control. The first is a state feedback result that assumes both attitude and angular velocity information. The second result removes the assumption that angular velocity is available. In

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addition, some heuristic rules for tuning the proposed strategies are given. In Sec. IV, we present simulation results that demonstrate our technique. In Sec. V, we offer some conclusions.

II. Background and Problem Statement

This section provides brief background material on unit quaternions. Complete discussions on the use of unit quaternions for attitude representation may be found in Refs. 17 and 18.

A unit quaternion is a four dimensional unit vector. A rotation of ϕ rad about a unit vector z is represented by the unit quaternion

$$p = \left(\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \atop p_0\right) = \begin{pmatrix} \hat{p} \\ p_0 \end{pmatrix} \triangleq \begin{Bmatrix} \sin(\phi/2)z \\ \cos(\phi/2) \end{Bmatrix}$$

where $\hat{p}$ is the vector part of the quaternion and p_0 is the scalar part of the quaternion.

The conjugate of a unit quaternion, which represents a rotation of $-\phi$ about z, is given by

$$p^* = \begin{pmatrix} -\hat{p} \\ p_0 \end{pmatrix}$$

Quaternion multiplication is defined by the equation

$$pq = \begin{pmatrix} q_0\hat{p} + p_0\hat{q} + \hat{p} \times \hat{q} \\ q_0 p_0 - \hat{q}^T\hat{p} \end{pmatrix}$$

The identity quaternion is given by $\mathbf{1} = (0, 0, 0, 1)^T$, where $pp^* = p^*p = \mathbf{1}$. On occasion we will have need to refer to the scalar and vector parts of the product of two or more quaternions. In that case we will use the notation $(pq)_0$ and $\widehat{pq}$ refer to the scalar and vector parts of pq, respectively. To simplify the equations in several places we will use the notation

$$\bar{\omega} = \begin{pmatrix} \omega \\ 0 \end{pmatrix}$$

If the attitude of the spacecraft is represented by q, then the equations of motion for the spacecraft are given by

$$\dot{q} = \tfrac{1}{2}q\bar{\omega} \tag{1}$$

$$J\dot{\omega} = -\omega \times J\omega + \tau \tag{2}$$

If q represents the attitude of the spacecraft, and q^d represents its desired attitude, then the error quaternion, which represents the attitude error between q and q^d, is given by $e \triangleq q^{d*}q$.

In this paper, the attitude control problem of rotating a rigid body from its current attitude q to a desired attitude q^d is considered, where we would like the instantaneous axis of rotation to align as closely as possible with an externally defined desired axis of rotation that is fixed in the inertial frame. Let the desired axis of rotation be represented by the unit vector u. A block diagram of the proposed approach is shown in Fig. 1, where both q^d and u are defined externally.

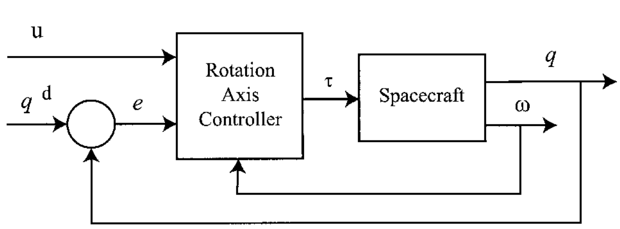


Fig. 1 Proposed block diagram of the closed-loop system.

III. Rotation Axis Control

In this section we present the main results. In Sec. III.A, we show how to factor a unit quaternion into two unit quaternions that represent a rotation about a fixed inertial axis u and a rotation about an axis perpendicular to u. In Sec. III.B, we derive a state feedback control law that minimizes the motion perpendicular to u. In Sec. III.C, we offer intuition about how the gains can be selected for a desired response. Finally, an output feedback result is presented in Sec. III.D.

A. Quaternion Factorization

Let $e = q^{d*}q$ be the error quaternion, and let u be an arbitrary desired axis of rotation fixed in the inertial frame. We define $e_\|$ as the unit quaternion that represents the portion of the rotation of e that is parallel to u, that is,

$$e_\|(u) = \begin{bmatrix} \sin(\theta/2)u \\ \cos(\theta/2) \end{bmatrix} \tag{3}$$

where

$$\theta = 2\tan^{-1}\left(\hat{e}^T u \big/ e_0\right) \tag{4}$$

Similarly, we can define the unit quaternion perpendicular to the desired axis of rotation as

$$e_\perp(u) = e_\|^* e = \begin{bmatrix} \cos(\theta/2)\hat{e} - e_0\sin(\theta/2)u - \sin(\theta/2)u \times \hat{e} \\ \cos(\theta/2)e_0 + \sin(\theta/2)\hat{e}^T u \end{bmatrix} \tag{5}$$

Lemma: Let e be the unit quaternion representing attitude error and let u be an arbitrarily defined desired axis of rotation, and let $e_\|(u)$ and $e_\perp(u)$ be defined as in Eqs. (3) and (5), respectively, then the following statements hold:

1) $e = e_\|(u)e_\perp(u)$.
2) $\hat{e}_\perp^T u = 0$.
3) $\hat{e}_\|^T \hat{e}_\perp = 0$.
4) The angular velocity $\omega_\| \triangleq \dot{\theta}u$ satisfies the rigid-body kinematic equation

$$\dot{e}_\| = \tfrac{1}{2}e_\|\bar{\omega}_\|$$

5) The angular velocity

$$\omega_\perp \triangleq \omega - \widehat{e_\perp^*\bar{\omega}_\| e_\perp}$$

satisfies the rigid-body kinematic equation

$$\dot{e}_\perp = \tfrac{1}{2}e_\perp\bar{\omega}_\perp$$

Proof: The proof of this lemma is based on standard quaternion manipulation. Full details appear in Ref. 19. □

B. State Feedback Rotation Axis Control

In this section, we will consider the control strategy given by

$$\tau = -k_1\hat{e} - d_1\omega - k_2\hat{e}_\perp - d_2(I - uu^T)\omega \tag{6}$$

where the first two term are similar to proportional-derivative (PD) control using quaternion feedback,[1] and the second two terms feed back the portion of the error quaternion that is perpendicular to u.

Theorem 1: Consider the spacecraft whose dynamics are given by Eqs. (1) and (2). Let u be an arbitrary unit vector representing the desired axis of rotation in the inertial frame, let $e_\perp$ be given by Eq. 5, and let the input torque be given by Eq. (6).

1) If the constants k_1, k_2, d_1, and d_2 are positive and $k_1 \neq k_2$, then $\hat{e}(t) \to 0$ and $\omega(t) \to 0$ asymptotically.

2) In addition, if

$$k_1\|e(0) - \mathbf{1}\|^2 + k_2\|e_\perp(0) - \mathbf{1}\|^2 + \tfrac{1}{2}\omega^T(0)J\omega(0) < 2k_1 + 2k_2$$

then $\|e(t) - \mathbf{1}\| \to 0$ and $\|\omega(t)\| \to 0$ asymptotically.

3) If condition 1 holds and the initial state satisfies

$$e(0) \neq -\mathbf{1}, \qquad \omega(0) = 0, \qquad \|e_\perp(0) - \mathbf{1}\| \leq \delta$$

$$\sqrt{\delta^2 + (k_1/k_2)\|e(0) - \mathbf{1}\|^2} \leq \epsilon \tag{7}$$

then $(q, \omega) \to (\mathbf{1}, 0)$ asymptotically, and $\|e_\perp(t) - \mathbf{1}\| \leq \epsilon$ for all $t \geq 0$.

The first statement of Theorem 1 guarantees regulation of the attitude error to zero. The second statement excludes the case of regulation of the error to $-\mathbf{1}$. The third statement guarantees that, if the component of the initial attitude error that is perpendicular to u is bounded by δ, throughout the maneuver it remains bounded by ϵ, where ϵ can be made arbitrarily close to δ by an appropriate choice of k_1 and k_2.

Proof: First consider the Lyapunov function candidate

$$V = k_1\|e - \mathbf{1}\|^2 + k_2\|e_\perp - \mathbf{1}\|^2 + \tfrac{1}{2}\omega^T J\omega$$

which is zero if and only if $e = \mathbf{1}$ and $\omega = 0$. Differentiating V, we obtain

$$\dot{V} = 2k_1(e - \mathbf{1})^T\dot{e} + 2k_2(e_\perp - \mathbf{1})\dot{e}_\perp + \omega^T J\dot{\omega}$$

Noting that $\dot{e} = q^{d*}\dot{q} = \frac{1}{2}q^{d*}q\bar{\omega} = \frac{1}{2}e\bar{\omega}$ and using the lemma and Eq. (2), we obtain

$$\dot{V} = k_1(e - \mathbf{1})^T e\bar{\omega} + 2k_2(e_\perp - \mathbf{1})e_\perp\bar{\omega}_\perp + \omega^T\tau$$

Observing that

$$(e - \mathbf{1})^T e\bar{\omega} = \begin{pmatrix}\hat{e}\\ e_0 - 1\end{pmatrix}^T \begin{pmatrix} e_0\omega + \omega \times \hat{e}\\ -\hat{e}^T\omega\end{pmatrix} = \omega^T\hat{e}$$

we get

$$\dot{V} = k_1\omega^T\hat{e} + k_2\omega_\perp^T\hat{e}_\perp + \omega^T\tau \tag{8}$$

When the definition of $\omega_\perp$ is used, it is straightforward to show that $\omega_\perp^T\hat{e}_\perp = \omega^T\hat{e}_\perp$. Therefore, $\dot{V}$ can be written as

$$\begin{aligned}\dot{V} &= k_1\omega^T\hat{e} + k_2\omega^T\hat{e}_\perp + \omega^T\tau\\ &= \omega^T(\tau + k_1\hat{e} + k_2\hat{e}_\perp)\\ &= \omega^T[-d_1\omega - d_2(I_3 - uu^T)\omega]\\ &= -\omega^T D\omega \end{aligned}\tag{9}$$

where $D = [d_1 I_3 + d_2(I_3 - uu^T)]$ and the third line was obtained from Eq. (6). Noting that D is symmetric and positive definite, we see that $\dot{V} \leq 0$.

Let $\Omega = \{(e, \omega) | \dot{V} = 0\}$ and let $\bar{\Omega}$ be the largest invariant subset in Ω. On $\bar{\Omega}$, $\omega(t) \equiv 0$ and $\tau(t) \equiv 0$; therefore, from Eqs. (2) and (6), we get that

$$k_1\hat{e} + k_2\hat{e}_\perp = 0 \tag{10}$$

Because

$$\hat{e} = e_{\perp 0}\hat{e}_\parallel + e_{\parallel 0}\hat{e}_\perp + \hat{e}_\perp \times \hat{e}_\parallel$$

substituting into Eq. (10) gives

$$k_1 e_{\perp 0}\hat{e}_\parallel + (k_2 + k_1 e_{\parallel 0})\hat{e}_\perp + k_1\hat{e}_\parallel \times \hat{e}_\perp = 0 \tag{11}$$

If we multiply Eq. 11 by $\hat{e}_\parallel^T$, $\hat{e}_\perp^T$, and $(\hat{e}_\perp \times \hat{e}_\parallel)^T$, respectively, we obtain the following relations:

$$e_{\perp 0}\|\hat{e}_\parallel\|^2 = 0 \tag{12}$$

$$(k_2 + k_1 e_{\parallel 0})\|\hat{e}_\perp\|^2 = 0 \tag{13}$$

$$\|\hat{e}_\perp \times \hat{e}_\parallel\|^2 = 0 \tag{14}$$

Note from the lemma that $\hat{e}_\perp^T\hat{e}_\parallel = 0$; then Eq. (14) indicates that $\hat{e}_\perp$ and $\hat{e}_\parallel$ are simultaneously parallel and perpendicular. Therefore, either $\hat{e}_\perp$ or $\hat{e}_\parallel$ must be zero. If $\hat{e}_\perp = 0$, then $e_{\perp 0} = \pm 1$, and from Eq. (12), $\hat{e}_\parallel = 0$. On the other hand, if $\hat{e}_\parallel = 0$, then $e_{\parallel 0} = \pm 1$, and provided that $k_1 \neq k_2$, Eq. (13) implies that $\hat{e}_\perp = 0$. Therefore, $\hat{e}_\parallel = \hat{e}_\perp = 0$, which implies that $e_{\parallel 0} = \pm 1$ and $e_{\perp 0} = \pm 1$, or equivalently $e = e_\parallel e_\perp = \pm\mathbf{1}$. Therefore, $\bar{\Omega} = \{(\pm\mathbf{1}, 0)\}$ and LaSalle's invariance principle ensures that $\hat{e}(t) \to 0$ and $\omega(t) \to 0$ asymptotically.

Second, suppose that $e = -\mathbf{1}$. It follows that $e_\parallel e_\perp = -\mathbf{1}$ or $e_\parallel = -e_\perp^*$. Therefore $\hat{e}_\parallel = \hat{e}_\perp$, and $e_{\parallel 0} = -e_{\perp 0}$. Because $\hat{e}_\parallel$ and $\hat{e}_\perp$ are simultaneously equal and perpendicular, they must both equal zero. From Eq. (4), $\hat{e} = 0$ implies that $\theta = 0$, which implies from Eq. 5 that $e_{\perp 0} = e_0 = -1$. Thus, we can see that $e = e_\perp = -\mathbf{1}$. Therefore, when $e = -\mathbf{1}$ and $\omega = 0$, the Lyapunov function is equal to $2k_1 + 2k_2$. Because $\dot{V} \leq 0$ if

$$V[e(0), \omega(0)] < 2k_1 + 2k_2$$

the attitude state can never reach the state $(e, \omega) = (-\mathbf{1}, 0)$, which implies that the attitude will converge to $(\mathbf{1}, 0)$ because this is the only remaining element of $\bar{\Omega}$.

Third, note that because $\omega(0) = 0$ and $q(0) \neq -\mathbf{1}$ condition 2 holds and, thus, $(q, \omega) \to (\mathbf{1}, 0)$ asymptotically. From Lyapunov theory, we get

$$\begin{aligned} k_2\|e_\perp(t) - \mathbf{1}\|^2 &\leq k_1\|e(t) - \mathbf{1}\|^2 + k_2\|e_\perp(t) - \mathbf{1}\|^2 + \tfrac{1}{2}\omega(t)^T J\omega(t)\\ &= V(t)\\ &\leq V(0)\\ &= k_1\|e(0) - \mathbf{1}\|^2 + k_2\|e_\perp(0) - \mathbf{1}\|^2 + \tfrac{1}{2}\omega(0)^T J\omega(0)\\ &= k_1\|e(0) - \mathbf{1}\|^2 + k_2\|e_\perp(0) - \mathbf{1}\|^2 \\ &\qquad [\text{because } \omega(0) = 0]\\ &\leq k_1\|e(0) - \mathbf{1}\|^2 + k_2\delta^2 \end{aligned}$$

from which we obtain

$$\|e_\perp(t) - \mathbf{1}\| \leq \sqrt{\delta^2 + (k_1/k_2)\|e(0) - \mathbf{1}\|^2} \leq \epsilon \qquad \square$$

C. Gain Selection Heuristics

The objective of this section is to offer some simple heuristics for choosing the gains k_1, d_1, k_2, and d_2. We begin by combining Eqs. (2) and (6) to obtain

$$J\dot{\omega} + \omega \times J\omega + d_1\omega + d_2(I_3 - uu^T)\omega + k_1\hat{e} + k_2\hat{e}_\perp = 0 \tag{15}$$

Projecting this equation onto the desired axis of rotation by multiplying both sides of Eq. (15) by u^T gives

$$u^T J\dot{\omega} + u^T(\omega \times J\omega) + d_1 u^T\omega + k_1 u^T\hat{e} = 0 \tag{16}$$

Note that k_2 and d_2, the gains associated with off-axis motion, do not appear in Eq. (16). Assuming that the off-axis motion is small, we can approximate $\hat{e}$ and ω as $\hat{e} \approx \sin(\theta/2)u$ and $\omega \approx \dot{\theta}u$. Substituting these approximations into Eq. (16) gives

$$u^T J u\ddot{\theta} + d_1\dot{\theta} + k_1\sin(\theta/2) = 0 \tag{17}$$

For small angles, $\sin(\theta/2) \approx \theta/2$, which allows us to write Eq. (17) as

$$\ddot{\theta} + (d_1/u^T J u)\dot{\theta} + (k_1/2u^T J u)\theta = 0 \tag{18}$$

which is a second-order linear system. The gains k_1 and d_1 can be selected to give the desired linear response for small angles near the axis of rotation.

Selection of k_2 should be made based on Eq. (7). Given ϵ, $\delta < \epsilon$, k_1, and $\|e(0) - \mathbf{1}\|$, k_2 is chosen such that

$$k_2 \geq k_1\frac{\|e(0) - \mathbf{1}\|}{\epsilon^2 - \delta^2}$$

The gain d_2 adds damping to the motion of $e_\perp$. Although the selection of d_2 appears arbitrary, note, however, from Eq. (6) that the matrix $d_1 I_3 + d_2(I_3 - uu^T)$ multiplies ω. Therefore, if $d_2 \gg d_1$, the condition number of this matrix becomes large, and small (possibly numeric) variations in ω may cause large swings in τ. Based on simulation, we have found that $10d_1 \leq d_2 \leq 100d_1$ results in adequate damping.

D. Output Feedback Rotation Axis Control

The control law given in Eq. (6) requires knowledge of both $\boldsymbol{e}$ and $\boldsymbol{\omega}$. This section uses the passivity ideas presented in Refs. 6 and 7 to remove the dependence on $\boldsymbol{\omega}$.

Theorem 2: Consider the spacecraft whose dynamics are given by Eqs. (1) and (2). Let $\boldsymbol{u}$ be an arbitrary unit vector representing the desired axis of rotation fixed in the inertial frame, let $\boldsymbol{e}_\perp$ be given by Eq. (5), and let the input torque be given by the following dynamic system:

$$\dot{\boldsymbol{\alpha}} = A\boldsymbol{\alpha} + B\boldsymbol{e}$$
$$\boldsymbol{y} = B^T PA\boldsymbol{\alpha} + B^T PB\boldsymbol{e}$$
$$\boldsymbol{\tau} = -k_1\hat{\boldsymbol{e}} - k_2\hat{\boldsymbol{e}}_\perp - \widehat{\boldsymbol{e}^*\boldsymbol{y}} \tag{19}$$

1) If $k_1 > 0$, $k_2 > 0$, $k_1 \neq k_2$, $A \in \mathbb{R}^{4\times 4}$ is Hurwitz, $B \in \mathbb{R}^{4\times 4}$ is full rank, and P is the positive definite solution to $A^T P + PA = -Q$, where $Q \in \mathbb{R}^{4\times 4}$ is negative definite, then $\hat{\boldsymbol{e}}(t) \to 0$ and $\boldsymbol{\omega}(t) \to 0$ asymptotically.

2) In addition, if $\boldsymbol{\alpha}(0) = -A^{-1}B\boldsymbol{e}(0)$ and

$$k_1\|\boldsymbol{e}(0) - \mathbf{1}\|^2 + k_2\|\boldsymbol{e}_\perp(0) - \mathbf{1}\|^2 + \tfrac{1}{2}\omega^T(0)J\omega(0) < 2k_1 + 2k_2$$

then $\|\boldsymbol{e}(t) - \mathbf{1}\| \to 0$ and $\|\boldsymbol{\omega}(t)\| \to 0$ asymptotically.

3) If condition 1 holds and the initial conditions satisfy

$$\boldsymbol{q}(0) \neq -\mathbf{1}, \qquad \boldsymbol{\alpha}(0) = -A^{-1}B\boldsymbol{e}(0), \qquad \boldsymbol{\omega}(0) = 0$$
$$\|\boldsymbol{e}_\perp(0) - \mathbf{1}\| \leq \delta, \qquad \sqrt{\delta^2 + (k_1/k_2)\|\boldsymbol{e}(0) - \mathbf{1}\|^2} \leq \epsilon$$

then $\|\boldsymbol{e}_\perp(t) - \mathbf{1}\| \leq \epsilon$ for all $t \geq 0$.

Proof: Note that the filter equation

$$\dot{\boldsymbol{\alpha}} = A\boldsymbol{\alpha} + B\boldsymbol{e}$$
$$\boldsymbol{y} = B^T PA\boldsymbol{\alpha} + B^T PB\boldsymbol{e}$$

can be expressed as

$$\dot{\bar{\boldsymbol{\alpha}}} = A\bar{\boldsymbol{\alpha}} + B\dot{\boldsymbol{e}}$$
$$\boldsymbol{y} = B^T P\bar{\boldsymbol{\alpha}} \tag{20}$$

where $\bar{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}}$.

Consider the Lyapunov function candidate

$$V = k_1\|\boldsymbol{e} - \mathbf{1}\|^2 + k_2\|\boldsymbol{e}_\perp - \mathbf{1}\|^2 + \tfrac{1}{2}\boldsymbol{\omega}^T J\boldsymbol{\omega} + \bar{\boldsymbol{\alpha}}^T P\bar{\boldsymbol{\alpha}}$$

Differentiating V, substituting in Eq. (20), and using the same steps as in the preceding section, we get

$$\begin{aligned}
\dot{V} &= \boldsymbol{\omega}^T(\boldsymbol{\tau} + k_1\hat{\boldsymbol{e}} + k_2\hat{\boldsymbol{e}}_\perp) - \bar{\boldsymbol{\alpha}}^T Q\bar{\boldsymbol{\alpha}} + 2\bar{\boldsymbol{\alpha}}^T PB\dot{\boldsymbol{e}} \\
&= \boldsymbol{\omega}^T(\boldsymbol{\tau} + k_1\hat{\boldsymbol{e}} + k_2\hat{\boldsymbol{e}}_\perp) - \bar{\boldsymbol{\alpha}}^T Q\bar{\boldsymbol{\alpha}} + \boldsymbol{y}^T(\boldsymbol{e}\boldsymbol{\omega}) \\
&= \boldsymbol{\omega}^T(\boldsymbol{\tau} + k_1\hat{\boldsymbol{e}} + k_2\hat{\boldsymbol{e}}_\perp) - \bar{\boldsymbol{\alpha}}^T Q\bar{\boldsymbol{\alpha}} + (\boldsymbol{y}^*\boldsymbol{e}\boldsymbol{\omega})_0 \\
&= \boldsymbol{\omega}^T(\boldsymbol{\tau} + k_1\hat{\boldsymbol{e}} + k_2\hat{\boldsymbol{e}}_\perp) - \bar{\boldsymbol{\alpha}}^T Q\bar{\boldsymbol{\alpha}} + \boldsymbol{\omega}^T\widehat{\boldsymbol{e}^*\boldsymbol{y}} \\
&= \boldsymbol{\omega}^T(\boldsymbol{\tau} + k_1\hat{\boldsymbol{e}} + k_2\hat{\boldsymbol{e}}_\perp + \widehat{\boldsymbol{e}^*\boldsymbol{y}}) - \bar{\boldsymbol{\alpha}}^T Q\bar{\boldsymbol{\alpha}} \\
&= -\bar{\boldsymbol{\alpha}}^T Q\bar{\boldsymbol{\alpha}} \\
&\leq 0
\end{aligned}$$

Let $\bar{\Omega}$ be the largest invariant subset of $\Omega = \{(\boldsymbol{e}, \boldsymbol{\omega}, \bar{\boldsymbol{\alpha}}) | \dot{V} = 0\}$. On $\bar{\Omega}$, $\bar{\boldsymbol{\alpha}}(t) \equiv 0$. This also implies that $\dot{\bar{\boldsymbol{\alpha}}}(t) \equiv 0$. From Eq. (20), on this set, $\dot{\boldsymbol{e}}(t) \equiv 0$. Because $\boldsymbol{\omega} = 2\boldsymbol{e}^*\dot{\boldsymbol{e}}$ and $\boldsymbol{\tau} = J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times J\boldsymbol{\omega}$, it also follows that $\boldsymbol{\omega}(t) \equiv \boldsymbol{\tau}(t) \equiv 0$. Furthermore, because $\boldsymbol{y} = B^T P\bar{\boldsymbol{\alpha}}$, on $\bar{\Omega}$, $\boldsymbol{y}(t) \equiv 0$. Therefore, $k_1\hat{\boldsymbol{e}} + k_2\hat{\boldsymbol{e}}_\perp = 0$. The arguments for the rest of the proof are similar to the proof of Theorem 1. □

IV. Simulation Results

In this section we present simulation results for a spacecraft whose model is given by Eqs. (1) and (2) where

$$J = \begin{pmatrix} 1200 & 100 & -200 \\ 100 & 2200 & 300 \\ -200 & 300 & 3100 \end{pmatrix}$$

Simulation results for four different controllers will be presented. The first controller is the state feedback, rotation axis controller given in Eq. (6), where $k_1 = 115$, $k_2 = 1150$, $d_1 = 736$, and $d_2 = 736$. The second controller is the output feedback, rotation axis controller given in Eq. (19), where $k_1 = 92$, $k_2 = 1150$, $A = -10I_4$, $B = 54I_4$, and

$$Q = 170I_4 - 100\begin{pmatrix}\boldsymbol{u}\\0\end{pmatrix}(\boldsymbol{u}^T\,0)$$

For comparison purposes, we will also present simulation result for the model-dependent eigenaxis controller presented in Ref. 12 and given by

$$\tau = -\boldsymbol{\omega} \times \hat{J}\boldsymbol{\omega} - k_1\hat{J}\hat{\boldsymbol{e}} - d_1\hat{J}\boldsymbol{\omega}$$

where $k_1 = 0.1$, $d_1 = 0.64$, and $\hat{J}$ is an approximation to J to within 10%. In addition, we will compare our results to the quaternion PD control suggested in Ref. 1, which is given by

$$\tau = -k_1\hat{\boldsymbol{e}} - d_1\boldsymbol{\omega}$$

where $k_1 = 100$ and $d_1 = 700$.

A. Example 1

The first maneuver that we will present is a rest-to-rest eigenaxis maneuver where the initial conditions of the spacecraft are given by

$$\boldsymbol{q}(0) = \begin{pmatrix} \sin\left(\dfrac{40\pi}{180}\right)\begin{bmatrix}1\\0\\0\end{bmatrix} \\ \cos\left(\dfrac{40\pi}{180}\right) \end{pmatrix}, \qquad \boldsymbol{\omega}(0) = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

and the desired attitude is given

$$\boldsymbol{q}^d = \begin{pmatrix} \sin(0)\begin{bmatrix}1\\0\\0\end{bmatrix} \\ \cos(0) \end{pmatrix}$$

In other words, the desired rest-to-rest maneuver is a 40-deg rotation about the axis $(1, 0, 0)^T$. The desired axis of rotation is also given by $\boldsymbol{u} = (1, 0, 0)^T$.

Figure 2 shows the simulation results. Note that the control gains for each of the four controllers have been tuned to give similar response for this particular maneuver. Figure 2a shows the convergence of the combined state metrics $\|\boldsymbol{e} - \mathbf{1}\| + \|\boldsymbol{\omega}\|$. Figure 2b shows the norm of the torque $\|\boldsymbol{\tau}\|$. Note that each control strategy requires roughly the same amount of torque for roughly the same convergence rate. Figure 2c plots $\|\boldsymbol{e}_\perp - \mathbf{1}\|$ and, therefore, shows the deviation of the attitude from the desired axis of rotation, which in this case is the eigenaxis at time zero. Note that because neither the model-dependent eigenaxis control strategy nor the PD control strategy have feedback on $\boldsymbol{e}_\perp$ small deviations from the desired axis of rotation are not compensated. Figure 2d shows a graph of

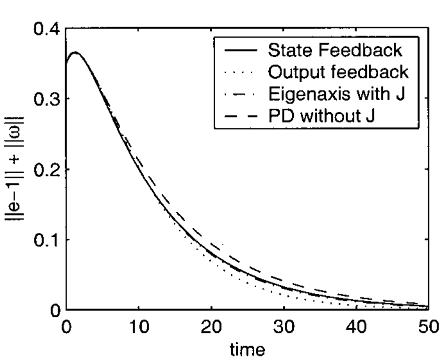
$$\left\| \frac{\hat{\boldsymbol{e}}(t)}{\|\hat{\boldsymbol{e}}(t)\|} - \boldsymbol{u} \right\|$$

Note that $\hat{\boldsymbol{e}}(t)/\|\hat{\boldsymbol{e}}(t)\|$ is the instantaneous rotation axis of the spacecraft with respect to the desired attitude. Therefore, Fig. 2d indicates the deviation of the instantaneous rotation axis from the desired axis of rotation. Figure 2d shows that the instantaneous rotation axis may deviate substantially from the original desired eigenaxis.

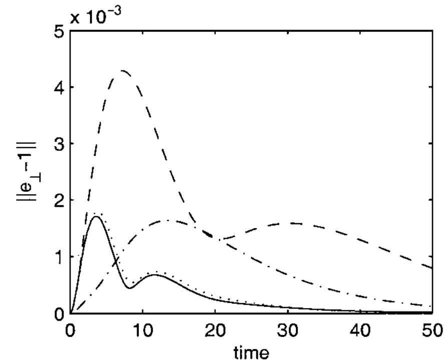
B. Example 2

For the second example, consider the same rest-to-rest maneuver presented in example 1, but where the desired rotation axis is given by

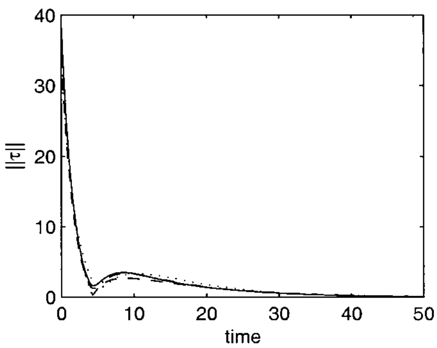
$$\boldsymbol{u} = \begin{pmatrix} \cos(30\pi/180) \\ \sin(30\pi/180) \\ 0 \end{pmatrix}$$



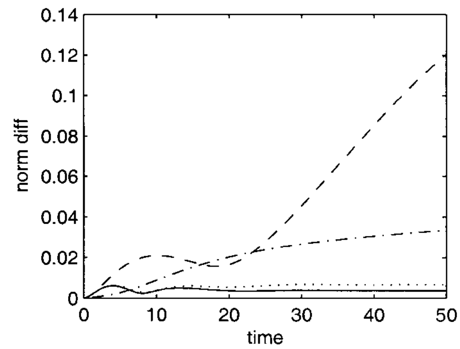
a) States $\|e - 1\| + \|\omega\|$



c) Error perpendicular to u^d: $\|e - 1\|$

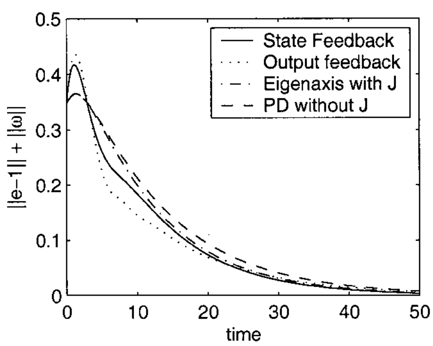


b) Control torque $\|\tau\|$

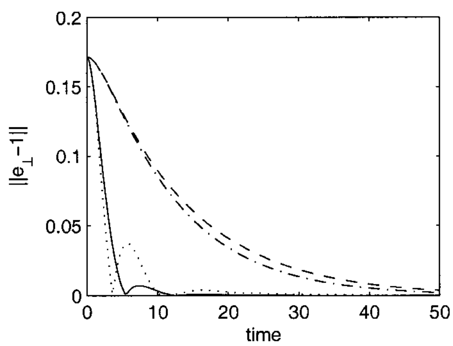


d) $\|e/\|e\| - u^d\|$

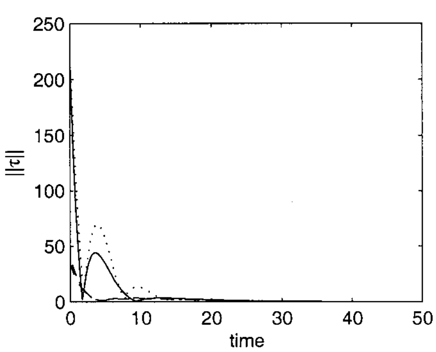
Fig. 2 Rest-to-rest maneuver about the eigenaxis.



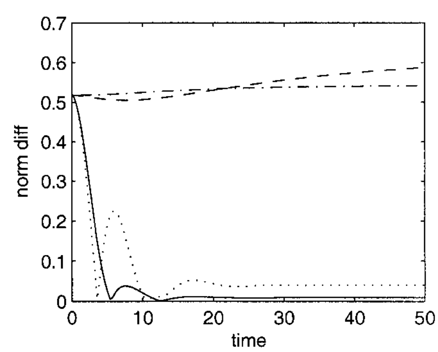
a) States $\|e - 1\| + \|\omega\|$



c) Error perpendicular to u^d: $\|e - 1\|$



b) Control torque $\|\tau\|$



d) $\|e/\|e\| - u^d\|$

Fig. 3 Rest-to-rest maneuver where desired rotation axis is 30 deg off of the eigenaxis at time zero.

In other words, the desired rotation axis is 30 deg off of the eigenaxis at time zero. Figure 3 shows the simulation results for this case.

Note, first, that the motion of the spacecraft perpendicular to u is driven to zero quickly by the proposed controllers. However, movement about a desired rotation axis as opposed to the "natural" eigenaxis comes at the price of control effort. The control torque is four time greater for the rotation axis controllers than the eigenaxis and PD control strategies. In example 1, the desired rotation axis was the natural eigenaxis and, therefore, the output of the four controllers was essentially the same. In this example, the rotation axis control strategies attempt to rotate the spacecraft to an orientation such that the remaining maneuver is aligned with the desired rotation axis. The alignment with the desired rotation axis requires extra bandwidth.

V. Conclusions

We have presented a model-independent attitude control law that is designed to align the instantaneous rotation axis of a spacecraft about an externally defined desired axis of rotation. Both state feedback and output feedback strategies were presented. The most obvious application of this approach is attitude formation control where a group of spacecraft must rotate about the same axis of rotation. The two simulation examples illustrate that the approach in fact regulates the instantaneous rotation axis to the desired axis of rotation. In addition, the examples illustrate that additional control effort is required to effect these types of maneuvers.

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